Dufour - Driven Generalized Mhd Double - Diffusive Convection

Hari Mohan, Pardeep Kumar

Department of Mathematics, International Centre for Distance Education and Open Learning (ICDEOL), Himachal Pradesh University, Summer Hill, Shimla-5, India

Corresponding author email: pkdureja@gmail.com

Abstract
In the present investigation the effect of uniform magnetic field on the Dufour-driven diffuse-convection of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid walls is considered. Some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude, are derived. The results for the horizontal layer geometry in the present case follow as a consequence.

Key words: Dufour-driven diffuse-convection; Rayleigh numbers; Prandtl numbers; Chandrasekhar number.

Introduction
Thermohaline convection or more generally double diffusive convection has matured into a subject possessing fundamental departure from its counterpart, namely single diffusive convection, and is of direct relevance in the fields of oceanography, astrophysics, limnology and chemical engineering etc. For a broad and a recent view of the subject one may be referred to Brandt and Fernando (Brandt and Fernando, 1966). The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour (Fitts DD, 1962; Groot and Mazur, 1962) effects. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon, known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The Thermohaline convection or more generally double diffusive convection has matured into a subject possessing fundamental departure from its counterpart, namely single diffusive convection, and is of direct relevance in the fields of oceanography, astrophysics, limnology and chemical engineering etc. analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and this is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. Recently, Mohan (Mohan, 1998) mollified the nastily behaving governing equations of Dufour-driven thermosolutal convection of the Veronis (Veronis, 1965) type by the construction of an appropriate linear transformation and derived the desired results concerning the linear growth rate and the behaviour of oscillatory motion on the lines suggested by Banerjee et. al (1981). Mohan (2010) investigated the problem of thermohaline convection coupled with cross-diffusions for the Veronis type configuration and derived a semi-circle theorem that prescribed upper limits for the complex growth rate of oscillatory motions of neutral or growing amplitude in such a manner that it naturally culminates in sufficient conditions precluding the non-existence of such motions.

Almost all the papers that are written on the subject are confined to horizontal layer geometry on account of complexity of the problem for arbitrary geometry. However, there do exist a class of results in the domain of hydrodynamic and hydromagnetic stability theory that possess the sparks of their generalization to containers of arbitrary shape (Sherman and Ostrach (1966)).

The present paper investigates the instability of Dufour-driven thermosolutal convection of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid walls in the presence of a uniform magnetic field applied in an arbitrary direction and derives some general qualitative results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude. The results for the horizontal layer geometry in the present case follow as a consequence.
Mathematical Formulation and Analysis

The relevant governing non-dimensional linearized perturbation equations in the present case with time dependence of the form exp (pt) (p = p_r + ip_i) are given by:

\[ \frac{P}{\sigma} \ddot{q} = -\nabla (p - \nabla \cdot \text{curl curl} \ddot{q}) + R_T \dot{\theta} - R_S \dot{\phi} + Q (\text{curl curl} \ddot{h}) \times \ddot{e} , \]

\[ (\nabla^2 - p) \ddot{h} + R_S \nabla^2 \dot{\phi} = -\ddot{q} \cdot \ddot{\beta} , \]

\[ (\tau \dot{\nabla}^2 - p) \ddot{\phi} = -\ddot{q} \cdot \ddot{\beta} , \]

\[ \text{curl curl} \ddot{h} + \frac{p \sigma}{\sigma} \ddot{h} = \text{curl} (\ddot{q} \times \ddot{e}) , \]

and \[ \nabla \cdot \ddot{q} = 0 = \nabla \cdot \ddot{h} . \]

In the above equations \( \ddot{q} = \text{q(x,y,z)}, \rho(x,y,z), \theta(x,y,z), \phi(x,y,z) \) and \( \ddot{h}(x,y,z) \) respectively denote the perturbed velocity, pressure, temperature, concentration and magnetic field and are complex valued functions defined on \( V, R_T = \frac{g \alpha \beta d^4}{K_0} \) is the thermal Rayleigh number, \( R_S = \frac{g \alpha' \beta' d^4}{K_0} \) is the concentration Rayleigh number, \( Q = \frac{\mu \epsilon H_0^2 d^2}{4 \pi \rho_0 \eta} \) is the Chandrasekhar number, \( \tau = \frac{\kappa'}{\kappa} \) is the Lewis number, \( V > 0 \) is referred here as Dufour number, and \( \ddot{\beta} \) is a unit vertical vector. Further, with \( d \) as the characteristic length, the equations have been cast into dimensionless forms by using the scale factors \( \frac{\kappa}{d}, \frac{d^2}{\kappa}, \beta d, \frac{\rho u \kappa}{d^2}, \beta d' \) and \( \frac{\kappa H_0}{\eta} \) for velocity, time, temperature, pressure, concentration and magnetic field respectively.

Associated with the system of equations (1)-(5) is a set of homogeneous and time independent boundary conditions. We shall limit our consideration to the region \( V \) completely confined by rigid walls, which may be thermal, and concentration-wise conducting or insulating and to see the case when the electrical conductivity of the wall is large in comparison to the field (see Sherman and Ostrach (8)). Thus we seek solutions of equations (1)-(5) in the simply connected subset \( V \) of \( R^3 \) subject to the following boundary conditions:

\[ \text{either } \ddot{q} = 0 = \theta = \phi = \hat{n} \times \text{curl} \ddot{h} \text{ on } S \quad (6) \]

\[ \text{or } \ddot{q} = 0 = \nabla \theta \cdot \hat{n} = \nabla \phi \cdot \hat{n} = \hat{n} \times \text{curl} \ddot{h} \text{ on } S . \quad (7) \]

where \( \hat{n} \) is a unit vector in the direction of the normal to boundary surface \( S \).

Equations (1)-(5) together with boundary conditions (6)-(7) constitute an eigen-value problem for \( p \) for given values of other parameters and the system is stable, neutral or unstable according as \( pr \) is negative, \( p_i \) is zero or positive. Further,

(a) \( p_i \neq 0 \) and \( p_r \geq 0 \) describe oscillatory motion of neutral or growing amplitude.

(b) \( R_T < 0, R_S < 0, \gamma' > 0 \) and \( Q = 0 \) describe Stern (1960) Thermohaline configuration in the present generalized set up which for convenience is epitomized in abbreviated form as GSTHC;

(c) \( \Gamma = \frac{|R_S|}{|R_T|} \) takes care of initial density gradient of the configuration.

Finally if \( p_i = 0 \Rightarrow p_r = 0 \), then the principle of exchange of stabilities (PES) is valid, otherwise, we have overstability.

We now, prove the following lemmas and theorems:

LEMMA 1: (Poincare : Inequality) – If \( f(x,y,z) \) is any smooth function which vanishes on \( S \) and \( \ell \) is the smallest distance between two parallel planes which just contains \( V \), then there exists a constant \( \lambda (> 2) \) such that

\[ \int_V |\nabla f|^2 dv \geq \lambda \frac{\ell^2}{V} \int|f|^2 dv \quad (8) \]

PROOF: See Joseph (10).

LEMMA 2: If \( (p, q, h, \theta, \phi) \) is a non-trivial solution of equation (1)-(5) together with either of the boundary conditions, then the following integral relations hold:
\[
\int \mathbf{q} \cdot \text{curl} \, \mathbf{q} \, dv = \int |\text{curl} \, \mathbf{q}|^2 \, dv.
\]
(9)

\[
\int \mathbf{q} \cdot \text{curl} \, (\mathbf{q} \times \hat{\mathbf{r}}) \, dv = \int \text{curl} \, (\mathbf{q} \times \hat{\mathbf{r}}) \cdot \text{curl} \mathbf{q} \, dv.
\]
(10)

\[
\int \mathbf{q} \cdot \text{curl} \, \text{curl} \, (\theta \hat{\mathbf{r}}) \, dv = 0 = \int \mathbf{q}^* \cdot \text{curl} \, \text{curl} \, (\phi \hat{\mathbf{r}}) \, dv.
\]
(11)

\[
\int \mathbf{q}^* \cdot \left[ \left( \text{curl} \, \mathbf{h} \right) \times \hat{\mathbf{r}} \right] \, dv = - \int \mathbf{h} \cdot \text{curl} \, \left( \mathbf{q}^* \hat{\mathbf{r}} \right) \, dv.
\]
(12)

\[
\int \mathbf{q}^* \cdot \left[ \hat{\mathbf{r}} \cdot \text{curl} \, \text{curl} \, \mathbf{h} \right] \, dv = - \int \text{curl} \, \mathbf{h} \cdot \text{curl} \left( \mathbf{q}^* \hat{\mathbf{r}} \right) \, dv.
\]
(13)

\[
\int \mathbf{h}^* \cdot \text{curl} \, \text{curl} \, \mathbf{h} \, dv = - \int \left| \text{curl} \, \mathbf{h} \right|^2 \, dv = \int \mathbf{h}^* \cdot \text{curl} \, \text{curl} \, \mathbf{h}^* \, dv,
\]
(14)

\[
\int \mathbf{q}^* \cdot \nabla (P) \, dv = 0.
\]
(15)

\[
\int \mathbf{q}^* \cdot \left[ \nabla \left( \text{div} \, \theta \hat{\mathbf{r}} \right) \right] \, dv = 0 = \int \mathbf{q}^* \cdot \left[ \nabla \left( \phi \hat{\mathbf{r}} \right) \right] \, dv.
\]
(16)

\[
\int \mathbf{q}^* \cdot \left[ \hat{\mathbf{r}} \cdot \text{curl} \, \mathbf{h} \right] \, dv = 0,
\]
(17)

\[
\int \theta^* \nabla^2 \theta \, dv = - \int \left| \nabla \theta \right|^2 \, dv = \int \theta^* \nabla \theta^* \, dv.
\]
(18)

and

\[
\int \phi^* \nabla^2 \phi \, dv = - \int \left| \nabla \phi \right|^2 \, dv = \int \phi^* \nabla \phi^* \, dv.
\]
(19)

Where \(^*\) denotes complex conjugate and \(|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}^*\) for any vector \(\mathbf{A}\).

**PROOF:** If \(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\) and \(\tilde{\mathbf{C}}\) are smooth vector-valued functions and \(\psi\) is a smooth scalar-valued function on \(V\) such that \(\mathbf{A} \times \tilde{\mathbf{B}}\) and \(\Psi \tilde{\mathbf{C}}\) vanish on \(S\), then using Gauss’ divergence theorem and the vector identities

\[\text{div} (\mathbf{A} \times \tilde{\mathbf{B}}) = \text{B} \cdot \text{curl} \, \mathbf{A} - \mathbf{A} \cdot \text{curl} \, \mathbf{B}\]

and

\[\text{div} (\Psi \tilde{\mathbf{C}}) = \nabla \Psi \cdot \mathbf{C} + \Psi \cdot \text{div} \, \tilde{\mathbf{C}}\]

it follows that

\[
\int \mathbf{B} \cdot \text{curl} \, \mathbf{A} \, dv = \int \mathbf{A} \cdot \text{curl} \, \mathbf{B} \, dv,
\]
(20)

and

\[
\int \nabla \Psi \cdot \mathbf{C} \, dv = - \int \Psi \, \text{div} \, \mathbf{C} \, dv.
\]
(21)

Now integral relations (9) – (14) follow from equation (20) by choosing \(\tilde{\mathbf{A}}\) and \(\tilde{\mathbf{B}}\) appropriately and integral relations (15)-(19) follow from equation (21) by choosing \(\Psi\) and \(\tilde{\mathbf{C}}\) appropriately.

This completes the proof of the lemma.
THEOREM 1: If \( \left(p, \tilde{q}, \tilde{h}, \theta, \phi\right) \), \( p = pr +ipi \), \( 0 < \gamma < \frac{\tau}{R_3} \) is a non-trivial solution of equations (1)-(5) together with either of the boundary conditions (6)-(7), \( R_T < 0, R_S < 0 \) and \( |R_S| \leq \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) \), then
\[ p_r = 0 \Rightarrow p_i \neq 0 \]

PROOF: Taking \( R_T = -|R_T|, R_S = -|R_S| \) and supposing \( p_r = 0 \Rightarrow p_i = 0 \) then \( p = 0 \), and therefore equations (1)-(5) become
\[ \nabla P + \text{curl} \left( \text{curl} \tilde{q} \right) = \left[-|R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) + |R_S| \right] \phi \hat{\beta} + Q \left( \text{curl} \tilde{h} \right) \times \hat{\ell}. \]  
(22)

\[ \nabla^2 \theta + R_3 \gamma^2 \phi = -\hat{q} \cdot \hat{k}, \]  
(23)

\[ \nabla^2 (\rho \theta) = -\tilde{q} \cdot \hat{k}, \]  
(24)

\[ \text{curl} \left( \text{curl} \tilde{h} \right) = \text{curl} \left( \tilde{q} \cdot \hat{\ell} \right). \]  
(25)

If \( \zeta = 0 - \tau \left(1 - \frac{R_3 \gamma}{\tau}\right) \phi \) then it follows from equations
\[ \nabla^2 \zeta = 0 \]  
(26)

Further, in view of boundary conditions (6)-(7), we have either
\( \zeta = 0 \) or \( \nabla \zeta \cdot \hat{n} = 0 \) on \( S \)  
(27)

The only solution of equation in \( V \) subject to either of the boundary condition in (27) is \( \zeta = 0 \). Consequently equation (22) assume the form
\[ \nabla P + \text{curl} \left( \text{curl} \tilde{q} \right) = \left[-|R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) + |R_S| \right] \phi \hat{\beta} + Q \left( \text{curl} \tilde{h} \right) \times \hat{\ell}. \]  
(28)

Taking dot product of equation (28) with \( \nabla^2 \hat{q} \), integrating the resulting equation over the domain \( V \) and using lemma 2, we get
\[ \int_{V} [\text{curl} \tilde{q}]^2 \, dv + Q \int \tilde{h} \cdot \text{curl}(\tilde{q} \times \hat{\ell}) \, dv = \left[|R_S| - \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) \right] \int_{V} \phi (\tilde{q} \cdot \hat{\beta}) \, dv \]  
(29)

Equation (29) upon using equation (24) and (25) and then appealing to lemma 2 yields the equation
\[ \int_{V} [\text{curl} \tilde{q}]^2 \, dv + Q \int |\text{curl} \tilde{h}|^2 \, dv = \left[|R_S| - \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) \right] \int_{V} |\nabla \phi|^2 \, dv. \]  
(30)

It follows from equation (30) that
\[ |R_S| > \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right), \]

a result contrary to the given hypothesis of the theorem. Hence \( p_r = 0 \Rightarrow p_i \neq 0 \). This completes the proof of the theorem.

Theorem 1, in the parlance of linear stability theory, may be stated as follows:

PES is not valid for the hydromagnetic GSTHC if
\[ |R_S| \leq \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right). \]

The following corollaries are direct consequences of Theorem 1

Cor. 1 – PES is not valid for GSTHV if \( |R_S| \leq \tau |R_T| \left(1 - \frac{R_3 \gamma}{\tau}\right) \).

Cor. 2 – PES is not valid for hydromagnetic initially top heavy GSTHC if \( \tau \left(1 - \frac{R_3 \gamma}{\tau}\right) > 1 \).

Cor. 3 – PES is not valid for initially top heavy GSTHC if \( \tau \left(1 - \frac{R_3 \gamma}{\tau}\right) > 1 \).
THEOREM 2 - If \( p, q, h, \theta, \phi \), \( p = pr + mpi \), is a non trivial solution of equations (1) – (5) with either of the boundary conditions (6) – (7), \( R_T < 0, R_S < 0 \) and \( \tau < \delta \leq 1 \), then for large \( Q \) (or for large \( |R_S| \) if \( Q = 0 \))

\[
pr \geq 0 \Rightarrow pi = 0,
\]

\[
\delta = \begin{cases} 1, & \text{if } R_S < 0 \text{ and } Q = 0 \\ \frac{\sigma}{\sigma_1}, & \text{if } R_S \leq 0 \text{ and } Q > 0. \end{cases}
\]

PROOF: Taking \( R_T = |R_T|, R_S = -|R_S| \) and using the transformations.

\[
\tilde{\theta} = \frac{1 - \tau}{R_S'} \theta + \phi
\]

\[
\tilde{\phi} = \phi
\]

\[
\tilde{q} = \hat{q}
\]

\[
\tilde{h} = \hat{h}
\]

equations (1) –(7) assume the forms

\[
P = -\nabla P - \text{curl curl} \ q\hat{q} - |R_T'\theta\hat{\beta} + |R_S'|\phi \cdot \hat{\beta} + Q(\text{curl} \ \hat{h}) \times \hat{e},
\]

(32)

\[
(\nabla^2 - p) \theta = -B\hat{q} \cdot \hat{\beta}.
\]

(33)

\[
(\nabla^2 - p) \phi = -\hat{q} \cdot \hat{\beta}.
\]

(34)

\[
\text{curl curl} \ \hat{h} + \frac{\rho_{\sigma_1}}{\sigma} \hat{h} = \text{curl} \ (\hat{q} \times \hat{e}),
\]

(35)

\[
\nabla \cdot \hat{q} = 0 \quad \nabla \cdot \hat{h}.
\]

(36)

With

\[
\tilde{q} = 0 = \theta = \hat{n} \times \text{curl} \ \tilde{h} \quad \text{on } S
\]

(37)

or

\[
\tilde{q} = 0 = \nabla \theta \cdot \hat{n} = \nabla \phi \cdot \hat{n} = \hat{n} \times \text{curl} \ \tilde{h} \quad \text{on } S
\]

(38)

where \( R_T = \frac{R_T R_S'}{1 - \tau} \), \( R_S = R_S + R_T' \), \( B = \left\{ 1 - \frac{1 - \tau}{R_S'} \right\} \) and the sign “-” has been omitted for simplicity. Operating on equation (32) by \( (\delta \text{ curl curl} + p) \) and using the vector identities

\[
\text{curl} \left( \Psi \tilde{A} \right) = \Psi \text{curl} A + \nabla \Psi + \tilde{A}
\]

\[
\text{curl} \left( \tilde{A} \times \tilde{B} \right) = (\tilde{B} \cdot \nabla) \tilde{A} - (\tilde{A} \cdot \nabla) \tilde{B} + \tilde{A} \text{ div } \tilde{B}
\]

and

\[
\nabla \left( \tilde{A} \times \hat{B} \right) = (\tilde{B} \cdot \nabla) \tilde{A} + (\tilde{A} \cdot \nabla) \tilde{B} + \tilde{B} \text{ curl } \tilde{A} + \tilde{A} \times \text{curl } \tilde{B},
\]

with an appropriate choice of \( \Psi, \tilde{A} \) and \( \hat{B} \), it follows that

\[
p\left( 1 + \frac{\delta}{\sigma} \right) \text{curl curl} \ q + \frac{P^2}{\sigma} \tilde{q} + p\nabla P
\]

\[
- |R_S'| \left\{ \delta \left[ \nabla (\text{div} \ \phi \hat{\beta}) - \nabla^2 \phi \hat{\beta} \right] + p \phi \hat{\beta} \right\}
\]

\[
- |R_T'\left[ \delta \left[ \nabla (\text{div} \ \theta \hat{\beta}) - \nabla^2 \theta \hat{\beta} \right] \right] +
\]

\[
+ Q \left[ \delta \nabla (\text{curl} \ \text{curl} \ \hat{h}) - \nabla (\text{curl} \ \text{curl} \ \hat{h}) - p \left( \text{curl} \ \hat{h} \times \hat{e} \right) \right]
\]

\[
= -\delta \text{ curl curl curl} \ q
\]

(39)
Taking the dot product of equation (39) with $\mathbf{q}^*$, integrating the resulting equation over the domain $\Omega$ and lemma 2, we have

$$
\begin{align*}
& \left(1 + \frac{\delta}{\sigma}\right) \int \mathbf{curl} \mathbf{q}^2 \, dv + \frac{\sigma^2}{\sigma} \int |\mathbf{q}|^2 \, dv - \int \mathbf{R}_1 \mathbf{1} \left( \nabla^2 \mathbf{q}^* \right) \cdot \hat{\mathbf{r}} \, dv \\
& \quad + \int \mathbf{R}_2 \left( \nabla^2 \mathbf{q}^* \right) \cdot \hat{\mathbf{r}} \, dv + p \int \mathbf{h} \cdot \mathbf{curl} \mathbf{q} \, dv \\
& \quad + Q \int \mathbf{h} \cdot \mathbf{curl} \mathbf{q} \, dv = -\delta \int \mathbf{q} \cdot \mathbf{curl} \mathbf{q} \cdot \mathbf{curl} \mathbf{q} \cdot dv
\end{align*}
$$

(40)

Since $Q$ (the ratio of magnetic to viscous forces) is very large, the effect of viscosity is thus significant near the bounding surfaces and in the above equation the integral on the right hand side (resulting from the viscous forces) is negligible in comparison with the last integral on the left hand side (resulting from the magnetic force) (cf. Sherman and Ostrach). Consequently, taking the right hand side of equation (40) to zero, eliminating $(\mathbf{q}^* \hat{\mathbf{r}})$ and $(\mathbf{q} \times \ell^*)$ from the resulting equation by using equation (33)-(35) and then appeal to lemma 2, we get

$$
\begin{align*}
& \left(1 + \frac{\delta}{\sigma}\right) \int \mathbf{curl} \mathbf{q}^2 \, dv + \frac{\sigma^2}{\sigma} \int |\mathbf{q}|^2 \, dv + |\mathbf{R}_1| \left( \delta \mathbf{q} \cdot \hat{\mathbf{r}} \right) + |\mathbf{R}_2| \left( \delta \mathbf{q} \cdot \hat{\mathbf{r}} \right) \\
& \quad + Q \left( \delta \mathbf{q} \cdot \mathbf{curl} \mathbf{q} \right) = 0
\end{align*}
$$

(41)

Equating the imaginary part of equation (41) to zero and assuming $p_i \neq 0$, we get

$$
\left(1 + \frac{\delta}{\sigma}\right) \int \mathbf{curl} \mathbf{q}^2 \, dv + \frac{2pr}{\sigma} \int |\mathbf{q}|^2 \, dv + |\mathbf{R}_1| \left( 1 - \delta \right) |\nabla \mathbf{q}|^2 \, dv \\
+ |\mathbf{R}_2| |\delta - r| |\nabla \mathbf{q}|^2 \, dv = 0
$$

Equation (42) cannot obviously be satisfied under the conditions of the theorem. Hence we must have

$p_i = 0$

This closes the proof of the theorem.

Theorem 2 implies that the hydromagnetic GSTHC on arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular PES is valid if $\tau \sigma_1 < \sigma < \sigma_1$. Further, this theorem also implies the validity of this result for the GSTHC if $\tau < 1$.

THEOREM 3: If $(p, \mathbf{q}, \mathbf{0}, \mathbf{\phi}, \mathbf{\hat{h}})$, $p = p_i + ip_\sigma$, $p_i \geq 0$, $p_i \neq 0$ is a non-trivial solution of equation (1)-(5) together with the boundary conditions (6) and $\mathbf{R}_1 < 0$, $\mathbf{R}_2 < 0$ and $\delta > 1$, then for large $Q$ (or for large $|\mathbf{R}_1|$ if $Q = 0$)

$$
|p| < \hat{\delta} \left[ |\mathbf{R}_1| (\delta - 1) B^2 + |\mathbf{R}_2| \right]
$$

where

$$
\hat{\delta} = \frac{l^2 \sigma}{\lambda (p + \delta)} , \quad \delta \text{ as in Theorem 2 and } \ell, \lambda \text{ are as in Lemma 1.}
$$

PROOF: It follows from equation (23) that

$$
\int \mathbf{curl} \mathbf{q} \cdot \mathbf{curl} \mathbf{q} \cdot dv = B^2 \int \mathbf{q} \cdot \hat{\mathbf{r}}^2 \, dv
$$

(43)
Equation (43) upon using lemma 2 gives
\[
\int \nabla^2 \psi^2 \, dv + 2 \text{pr} \int \nabla \psi^2 \, dv + |\psi|^2 \int | \psi |^2 \, dv = \int [\hat{q} \cdot \hat{\beta}]^2 \, dv
\]  
(44)

Equation (44), upon using \( \text{pr} \geq 0 \), \( \text{pi} \neq 0 \) gives
\[
\int |\psi|^2 \, dv < \frac{B^2}{|p|} \int [\hat{q} \cdot \hat{\beta}]^2 \, dv \leq \frac{B^2}{|p|} \int |\hat{q}|^2 \, dv
\]  
(45)

Again multiplying (33) by \( \theta^* \), integrating over the domain \( V \), using lemma 2 and equating the real parts of the resulting equation, we have
\[
\int |\nabla \psi|^2 \, dv + \text{pr} \int |\psi|^2 \, dv = \text{Real part of } \left( B \cdot \int \theta^* [\hat{q} \cdot \hat{\beta}] \, dv \right)
\]
\[
\leq \left| B \cdot \int \theta^* [\hat{q} \cdot \hat{\beta}] \, dv \right| \leq B \cdot \int |\theta| |\hat{q} \cdot \hat{\beta}| \, dv
\]
which upon using Schwartz’s inequality and the fact that \( \text{pr} \geq 0 \), gives
\[
\int |\nabla \psi|^2 \, dv \leq B \left[ \int |\psi|^2 \, dv \right]^{\frac{1}{2}} \left[ \left( B \cdot \int |\hat{q}|^2 \, dv \right) \right]^{\frac{1}{2}}
\]
\[
\leq B \left[ \int |\theta|^2 \, dv \right]^{\frac{1}{2}} \left[ \left( B \cdot \int |\hat{q}|^2 \, dv \right) \right]^{\frac{1}{2}}
\]  
(46)

Combining inequalities (45) and (46), we get
\[
\int |\nabla \psi|^2 \, dv \leq \frac{B \cdot B}{|p|} \int |\theta|^2 \, dv.
\]  
(47)

Further, the solenoidal character of the velocity field \( \hat{q} \) namely \( \text{div} \ \hat{q} = 0 \), implies that
\[
\int \text{curl} \ \hat{q}^2 \, dv = \int \left( \hat{q}^* \cdot \text{curl} \ \text{curl} \ \hat{q} \right) \, dv = -\int \hat{q}^* \nabla^2 \hat{q} \, dv
\]
which upon taking \( \hat{q} = (u, v, w) \), gives
\[
\int \text{curl} \ \hat{q}^2 \, dv = \int \left( |\nabla U|^2 + |\nabla V|^2 + |\nabla W|^2 \right) \, dv
\]  
(48)

Equation (48) together with lemma 1 yields the inequality
\[
\int |\hat{q}| \, dv < \frac{\ell^2}{\lambda} \int \text{curl} \ \hat{q}^2 \, dv
\]  
(49)

Inequality (47) and (49) implies that
\[
\int |\nabla \psi|^2 \, dv \leq \frac{B^2 \ell^2}{\lambda |p|} \int \text{curl} \ \hat{q}^2 \, dv
\]  
(50)

Similarly proceeding from equation (34), and emulating the steps in the derivation of inequality (50), we have
\[
\int |\nabla \psi|^2 \, dv \leq \frac{\ell^2}{\lambda |p|} \int \text{curl} \ \hat{q}^2 \, dv
\]  
(51)

Using inequality (5) and (51) in equation (42), we get
\[
\left( \frac{\sigma + \delta}{\sigma} \right) \left[ p - \delta \left( R^* + (\delta - 1)B^2 + |R^*_S| \right) \right] \int \text{curl} \ \hat{q}^2 \, dv
\]
\[
+ \frac{2|p|}{\sigma} \int |\hat{q}|^2 \, dv + \delta |R^*_S| \int |\nabla \psi|^2 \, dv < 0.
\]  
(52)

Inequality (52) clearly implies that
This completes the proof of the theorem.

Theorem 3 implies that the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for the hydromagnetic GSTHC lies inside a semi-circle with centre origin and radius:

$$\text{Radius} = \hat{\delta} \left| R_T' \right| (\delta - 1) B^2 + \left| R_S' \right| \left( \delta = \frac{\sigma}{\sigma_1} \right)$$

in the right half of the complex p-plane.

Conclusion

The present paper investigates the instability of Dufour–driven thermosolutal convection of a fluid completely confined in an arbitrary region bounded by rigid walls in the presence of a uniform magnetic field applied in an arbitrary direction. It has been found that Principle of exchange of stabilities is not valid for the hydromagnetic generalized Stern’s thermohaline configuration if $|R_S| \leq \tau \left| R_T' \right| \left( 1 - \frac{R_S \gamma}{\tau} \right)$. Secondly, for large Chandrasekhar number and $\tau < \delta < 1$, a neutral or unstable mode is definitely non oscillatory in character and in particular PES is valid. Finally the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable lies inside a semi-circle with centre origin and radius $\text{Radius} = \hat{\delta} \left| R_T' \right| (\delta - 1) B^2 + \left| R_S' \right|$ in the right half of the complex p-plane. Further, the results for the horizontal layer geometry in case of single-diffusive or double diffusive fluids follow as a consequence by taking $\lambda = \pi^2, I = 1$ respectively.

References


